

# MAXIMAL 0-1-FILLINGS OF MOON POLYOMINOES WITH RESTRICTED CHAIN LENGTHS AND RC-GRAPHS

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**ABSTRACT.** We show that maximal 0-1-fillings of moon polyominoes with restricted chain lengths can be identified with certain rc-graphs, also known as pipe dreams. In particular, this exhibits a connection between maximal 0-1-fillings of Ferrers shapes and Schubert polynomials. Moreover, it entails a bijective proof showing that the number of maximal fillings of a stack polyomino  $S$  with no north-east chains longer than  $k$  depends only on  $k$  and the multiset of column heights of  $S$ .

Our main contribution is a slightly stronger theorem, which in turn leads us to conjecture that the poset of rc-graphs with covering relation given by generalised chute moves is in fact a lattice.

## 1. INTRODUCTION

**1.1. Triangulations, multitriangulations and 0-1-fillings.** The systematic study of 0-1-fillings of polyominoes with restricted chain lengths likely originates in an article by Jakob Jonsson [5]. At first, he was interested in a generalisation of triangulations, where the objects under consideration are maximal sets of diagonals of the  $n$ -gon, such that at most  $k$  diagonals are allowed to cross mutually. Thus, in the case  $k = 1$  one recovers ordinary triangulations. He realised these objects as fillings of the staircase shaped polyomino with row-lengths  $n - 1, n - 2, \dots, 1$  with zeros and ones. The condition that at most  $k$  diagonals cross mutually then translates into the condition that the longest north-east chain in the filling has length  $k$ , see Definition 2.3. Instead of studying fillings of the staircase shape only, he went on to consider more general shapes which he called *stack* and *moon polyominoes*, see Definition 2.2 and Figure 1.

For stack polyominoes he was able to prove that the number of maximal fillings depends only on  $k$  and the multiset of heights of the columns, not on the particular shape of the polyomino. He conjectured

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*Key words and phrases.* multitriangulations, rc-graphs, Edelman-Greene insertion, Schubert polynomials.

that this statement holds more generally for moon polyominoes, which was eventually proved by the author [13] using a technique introduced by Christian Krattenthaler [8] based on Sergey Fomin’s growth diagrams for the Robinson-Schensted-Knuth correspondence. However, the proof given there is not fully bijective: what one would hope for is a correspondence between fillings of any two moon polyominoes that differ only by a permutation of the columns. This article is a step towards this goal.

**1.2. RC-graphs and the subword complex.** RC-graphs (for ‘reduced word compatible sequence graphs’, see [1], also known as ‘pipe dreams’ see [7]) were introduced by Sergey Fomin and Anatol Kirillov [3] to prove various properties of Schubert polynomials. Namely, for a given permutation  $w$ , the Schubert polynomial  $\mathfrak{S}_w$  can be regarded as the generating function of rc-graphs, see Remark 4.

A different point of view is to consider them as facets of a certain simplicial complex. Let  $w_0$  be the long permutation  $n \cdots 21$ , and consider its reduced factorisation

$$Q = s_{n-1} \cdots s_2 s_1 \ s_{n-1} \cdots s_3 s_2 \ \cdots \cdots \ s_{n-1} s_{n-2} \ s_{n-1}.$$

Then the subword complex associated to  $Q$  and  $w$  introduced by Allen Knutson and Ezra Miller [7, 6] has as facets those subwords of  $Q$  that are reduced factorisations of  $w$ . Subword complexes enjoy beautiful topological properties, which are transferred by the main theorem of this article to the simplicial complex of 0-1-fillings, as observed by Christian Stump [17], see also the article by Luis Serrano and Christian Stump [15].

The intimate connection between maximal fillings and rc-graphs demonstrated by the main theorem of this article, Theorem 3.2, *should* not have come as a surprise. Indeed, Sergey Fomin and Anatol Kirillov [4] established a connection between reduced words and reverse plane partitions already thirteen years ago, which is not much less than the case of Ferrers shapes in Theorem 4.3. They even pointed towards the possibility of a bijective proof using the Edelman-Greene correspondence.

More recently, the connection between Schubert polynomials and triangulations was noticed by Alexander Woo [18]. Vincent Pilaud and Michel Pocchiola [11] discovered rc-graphs (under the name ‘beam arrangements’) more generally for multitriangulations, however, they were unaware of the theory of Schubert polynomials. In particular, Theorem 3.18 of Vincent Pilaud’s thesis [10] (see also Theorem 21 of [11]) is a variant of our Theorem 3.2 for multitriangulations.

Finally, Christian Stump and the author of the present article became aware of an article by Vincent Pilaud and Francisco Santos [12] that describes the structure of multitriangulations in terms of so-called  $k$ -stars (introduced by Harold Coxeter). We then decided to translate this concept to the language of fillings, and discovered pipe dreams yet again.

## 2. DEFINITIONS

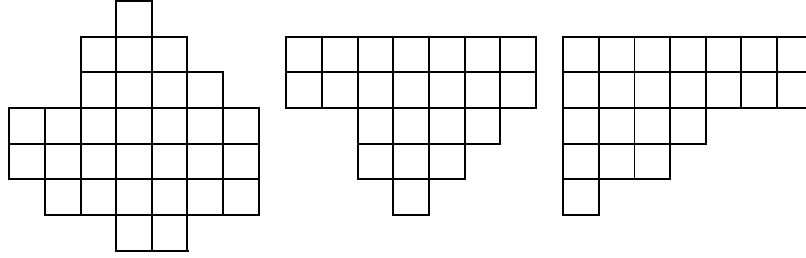


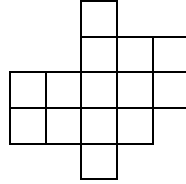
FIGURE 1. a moon-polyomino, a stack-polyomino and a Ferrers diagram

### 2.1. Polyominoes.

**Definition 2.1.** A *polyomino* is a finite subset of the quarter plane  $\mathbb{N}^2$ , where we regard an element of  $\mathbb{N}^2$  as a cell. A *column* of a polyomino is the set of cells along a vertical line, a *row* is the set of cells along a horizontal line. We are using ‘English’ (or matrix) conventions for the indexing of the rows and columns of polyominoes: the top row and the left-most column have index 1.

The polyomino is *convex*, if for any two cells in a column (rsp. row), the elements of  $\mathbb{N}^2$  in between are also cells of the polyomino. It is *intersection-free*, if any two columns are *comparable*, i.e., the set of row coordinates of cells in one column is contained in the set of row coordinates of cells in the other. Equivalently, it is intersection-free, if any two rows are comparable.

For example, the polyomino



is convex, but not intersection-free, since the first and the last columns are incomparable.

**Definition 2.2.** A *moon polyomino* (or L-convex polyomino) is a convex, intersection-free polyomino. Equivalently we can require that any two cells of the polyomino can be connected by a path consisting of neighbouring cells in the polyomino, that changes direction at most once. A *stack polyomino* is a moon-polyomino where all columns start at the same level. A *Ferrers diagram* is a stack-polyomino with weakly decreasing row widths  $\lambda_1, \lambda_2, \dots, \lambda_n$ , reading rows from top to bottom.

Because a moon-polyomino is intersection free, the set of rows of maximal length in a moon polyomino must be consecutive. We call the set of rows including these and the rows above the *top half* of the polyomino. Similarly, the set of columns of maximal length, and all columns to the right of these, is the *right half* of the polyomino. The intersection of the top and the right half is the *top right quarter* of  $M$ .

## 2.2. Fillings and Chains.

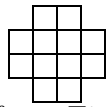
**Definition 2.3.** A 0-1-*filling* of a polyomino is an assignment of numbers 0 and 1 to the cells of the polyomino. Cells containing 0 are also called *empty*.

A *north-east chain* is a sequence of non-zero entries in a filling such that the smallest rectangle containing all its elements is completely contained in the moon polyomino and such that for any two of its elements one is strictly to the right and strictly above the other.

As it turns out, it is more convenient to draw dots instead of ones and leave cells filled with zeros empty. Two examples of (rather special) fillings of a moon polyomino are depicted in Figure 6. In both examples the length of the longest north-east chain is 2.

**Definition 2.4.**  $\mathcal{F}_{01}^{ne}(M, k)$  is the set of 0-1-fillings of the moon polyomino  $M$  whose longest north-east chain has length  $k$  and that are *maximal*, i.e., assigning an empty cell a 1 would create a north-east chain of length  $k + 1$ . For a vector  $\mathbf{r}$  of integers,  $\mathcal{F}_{01}^{ne}(M, k, \mathbf{r})$  is the subset of  $\mathcal{F}_{01}^{ne}(M, k)$  consisting of those fillings that have exactly  $\mathbf{r}_i$  zero entries in row  $i$ .

For any filling in  $\mathcal{F}_{01}^{ne}(M, k)$ , and an empty cell  $\epsilon$ , there must be a chain  $C$  such that replacing the 0 with 1 in  $\epsilon$ , and adding  $\epsilon$  to  $C$ , would make  $C$  into a  $(k + 1)$ -chain. In this situation, we say that  $C$  is a *maximal chain* for  $\epsilon$ .

For example, when  $M$  is the moon polyomino , the set  $\mathcal{F}_{01}^{ne}(M, 1)$  consists of ten fillings, as can be inferred from Figure 4.

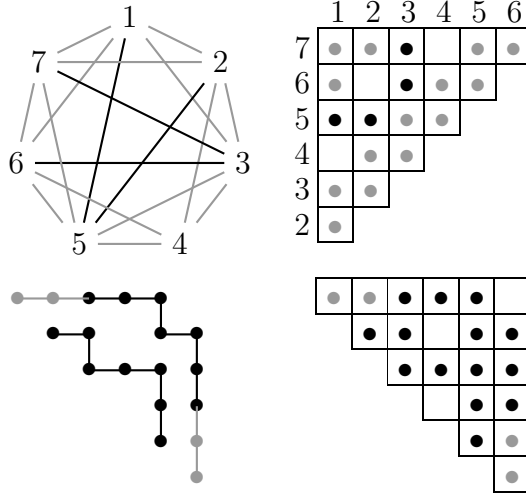


FIGURE 2. a 2-triangulation with corresponding filling of the staircase  $\lambda_0$  and a fan of two Dyck paths with corresponding filling of the reverse staircase  $\lambda_0^{rev}$ .

*Remark 1.* Note that extending the first  $k$  rows and columns of a Ferrers diagram does not affect the set  $\mathcal{F}_{01}^{ne}$ , which is why we choose to fix the number of zero entries instead of entries equal to 1, although the latter might seem more natural at first glance.

*Remark 2.* For the staircase shape  $\lambda_0$  with  $n-1$  rows the set  $\mathcal{F}_{01}^{ne}(\lambda_0, k)$  has a particularly beautiful interpretation, namely as the set of  $k$ -triangulations of the  $n$ -gon. More precisely, label the vertices of the  $n$ -gon clockwise from 1 to  $n$ , and identify a cell of the shape in row  $i$  and column  $j$  with the pair  $(n-i+1, j)$  of vertices. Thus, the entries in the filling equal to 1 define a set of diagonals of the  $n$ -gon. It is not hard to check that a north-east chain of length  $k$  in the filling corresponds to a set of  $k$  mutually crossing diagonals in the  $n$ -gon.

Maximal fillings of the reverse staircase shape  $\lambda_0^{rev}$  for a given  $k$  are in bijection with fans of  $k$  Dyck paths. An illustration of both correspondences is given in Figure 2. These correspondences were Jakob Jonsson's [5] starting point to prove (in a quite non-bijective fashion) that there are as many  $k$ -triangulations of the  $n$ -gon as fans of  $k$  non-intersecting Dyck paths with  $n-2k$  up steps each. Luis Serrano and Christian Stump [15] provided the first completely bijective proof of this fact, which we generalise in Section 4. Remarkably, Alex Woo [18] used the same methods already much earlier for the case of triangulations and Dyck paths, *i.e.*,  $k=1$ .

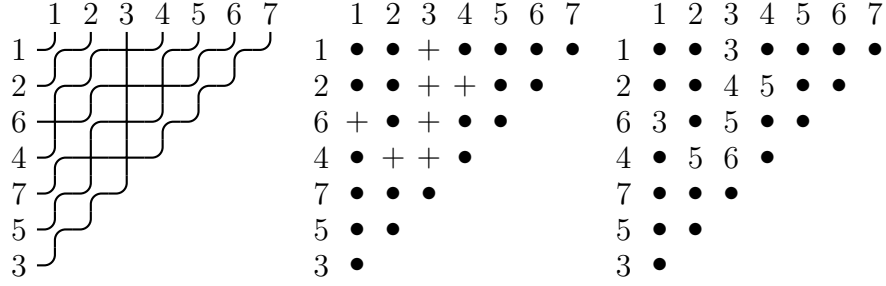


FIGURE 3. the reduced pipe dream associated to the reduced factorisation  $s_3s_5s_4s_5s_3s_6s_5$  of  $1, 2, 6, 4, 7, 5, 3$ .

**2.3. Pipe dreams.** In this section we collect some results around pipe dreams and rc-graphs. All of these statements can be found in [1] together with precise references.

**Definition 2.5.** A *pipe dream* for a permutation  $w$  is a filling of a the quarter plane  $\mathbb{N}^2$ , regarding each element of  $\mathbb{N}^2$  as a cell, with *elbow joints*  $\curvearrowright$  and a finite number of *crosses*  $+$ , such that a pipe entering from above in column  $i$  exits to the left from row  $w^{-1}(i)$ . A pipe dream is *reduced* if each pair of pipes crosses at most once, it is then also called *rc-graph*.  $\mathcal{RC}(w)$  is the set of reduced pipe dreams for  $w$ , and, for a vector  $\mathbf{r}$  of integers,  $\mathcal{RC}(w, \mathbf{r})$  is the subset of  $\mathcal{RC}(w)$  having precisely  $\mathbf{r}_i$  crosses in row  $i$ .

Usually it will be more convenient to draw dots instead of elbow joints and sometimes to omit crosses. We will do so without further notice.

*Remark 3.* We can associate a reduced factorisation of  $w$  to any pipe dream in  $\mathcal{RC}(w)$  as follows: replace each cross appearing in row  $i$  and column  $j$  of the pipe dream with the elementary transposition  $(i + j - 1, i + j)$ . Then the reduced factorisation of  $w$  is given by the sequence of transpositions obtained by reading each row of the pipe dream from right to left, and the rows from top to bottom. An example can be found in Figure 3, where we write  $s_i$  for the elementary transposition  $(i, i + 1)$ .

*Remark 4.* Using reduced pipe dreams, it is possible to define the Schubert polynomial  $\mathfrak{S}_w$  for the permutation  $w$  in a very concrete way. For a reduced pipe dream  $D \in \mathcal{RC}(w)$ , define  $x^D = \prod_{(i,j) \in D} x_i$ , where the product runs over all crosses in the pipe dream. Then the Schubert

polynomial is just the generating function for pipe dreams:

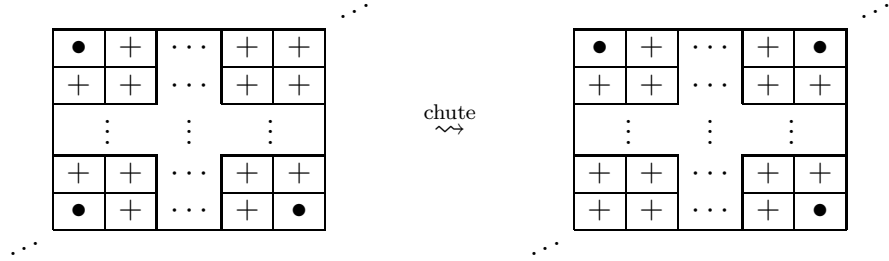
$$\mathfrak{S}_w = \sum_{D \in \mathcal{RC}(w)} x^D.$$

This definition of Schubert polynomials and their evaluation by Sergey Fomin and Anatol Kirillov [4] was used by Christian Stump [17] to give a simple proof of the product formula for the number of  $k$ -triangulations of the  $n$ -gon

$$\prod_{1 \leq i, j < n-2k} \frac{i + j + 2k}{i + j}.$$

We now define an operation on pipe dreams which was introduced in a slightly less general form by Nantel Bergeron and Sara Billey [1]. It will be the main tool in the proof of Theorem 3.2.

**Definition 2.6.** Let  $D \in \mathcal{RC}(w)$  be a pipe dream. Then a *chute move* is a modification of  $D$  of the following form:



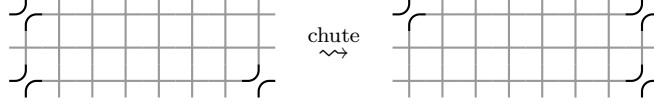
More formally, a *chutable rectangle* is a rectangular region  $r$  inside a pipe dream  $D$  with at least two columns and two rows such that all but the following three locations of  $r$  are crosses: the north-west, south-west, and south-east corners. Applying a *chute move* to  $D$  is accomplished by placing a  $+$  in the south-west corner of a chutable rectangle  $r$  and removing the  $+$  from the north-east corner of  $r$ . We call the inverse operation *inverse chute move*.

The following lemma was given by Nantel Bergeron and Sara Billey [1, Lemma 3.5] for two rowed chute moves, the proof is valid for our generalised chute moves without modification:

**Lemma 2.7.** *The set  $\mathcal{RC}(w)$  of reduced pipe dreams for  $w$  is closed under chute moves.*

*Proof.* The pictorial description of chute moves in terms of pipes immediately shows that the permutation associated to the pipe dream remains unchanged. For example, here is the picture associated with a

three rowed chute move:



□

*Remark 5.* It follows that chute moves define a partial order on  $\mathcal{RC}(w)$ , where  $D$  is covered by  $E$  if there is a chute move transforming  $E$  into  $D$ . Nantel Bergeron and Sara Billey restricted their attention to two rowed chute moves. For this case, their main theorem states that the poset defined by chute moves has a unique maximal element, namely

$$D_{top}(w) = \{(c, j) : c \leq \#\{i : i < w_j^{-1}, w_i > j\}\}.$$

It is easy to see that considering general chute moves, the poset has also a unique minimal element, namely

$$D_{bot}(w) = \{(i, c) : c \leq \#\{j : j > i, w_j < w_i\}\}.$$

In the next section we will show a statement similar in spirit to the main theorem of Nantel Bergeron and Sara Billey for the more general chute moves defined above.

After generating and analysing some of these posets using **Sage** [14], see Figure 4 for an example, we became convinced that they should have much more structure:

**Conjecture 2.8.** *The poset of reduced pipe dreams defined by (general) chute moves is in fact a lattice.*

There is another natural way to transform one reduced pipe dream into another, originating in the concept of flipping a diagonal of a triangulation. Namely, consider an elbow joint in the pipe dream. Since any pair of pipes crosses at most once, there is at most one location where the pipes originating from the given elbow joint cross. If there is such a crossing, replace the elbow joint by a cross and the cross by an elbow joint. Clearly, the result is again a reduced pipe dream, associated to the same permutation.

It is believed (see Vincent Pilaud and Michel Pocchiola [11], Question 51) that the simplicial complex of multitrangulations can be realised as a polytope, in this case the graph of flips would be the graph of the polytope. Note that the graph of chute moves is a subgraph of the graph of flips. Is Conjecture 2.8 related to the question of polytopality?



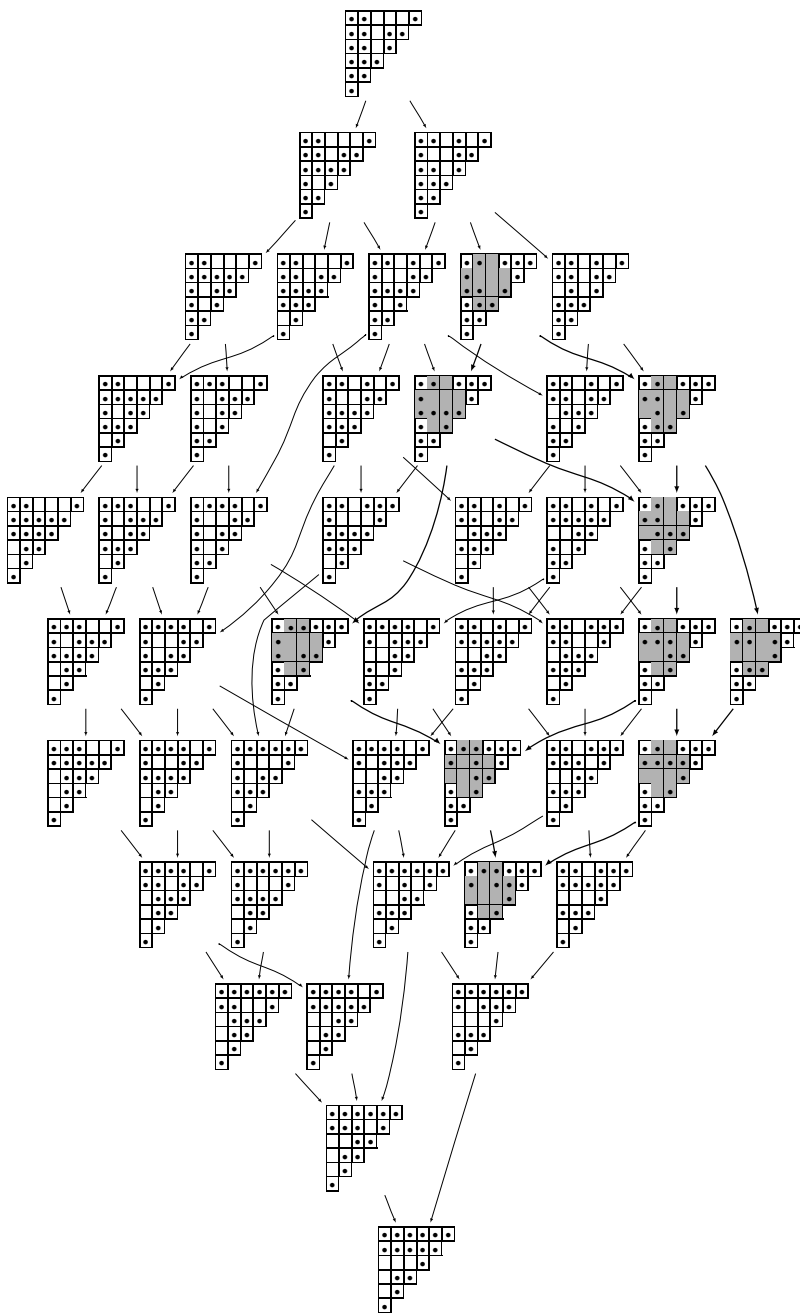



FIGURE 4. the poset of reduced pipe dreams for the permutation 1, 2, 6, 4, 5, 3. The interval of 0-1-fillings with  $k = 1$  of the moon polyomino  is emphasised.

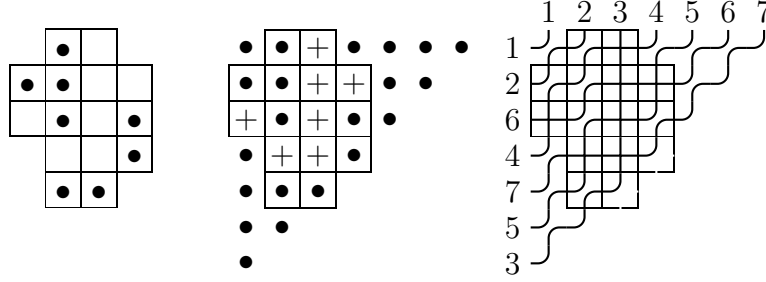


FIGURE 5. a maximal filling and the associated pipe dream.

### 3. MAXIMAL FILLINGS OF MOON POLYOMINOES AND PIPE DREAMS

Consider a maximal filling in  $\mathcal{F}_{01}^{ne}(M, k)$ . Recall that we regard a moon polyomino  $M$  as a finite subset of  $\mathbb{N}^2$ . Also, recall that a pipe dream is nothing but a filling of  $\mathbb{N}^2$  with elbow joints and a finite number of crosses. Thus, replacing zeros in the filling of the moon polyomino with crosses, and all cells in the filling containing ones as well as all cells not in  $M$  with elbow joints, we clearly obtain a pipe dream for some permutation  $w$ . An example of this transformation is given in Figure 5. We will see in this section that the pipe dreams obtained in this way are in fact reduced.

One may notice that the permutation associated with the pipe dream so constructed depends somewhat on the embedding of the polyomino into the quarter plane. Although one can check that this dependence is not substantial for what is to follow, we will assume for simplicity that the top row and the left-most column of the polyomino have index 1 and indices increase from top to bottom and from left to right.

Even without the knowledge that the pipe dream is reduced we can speak of chute moves applied to fillings in  $\mathcal{F}_{01}^{ne}(M, k)$ . However, a priori it is not clear under which conditions the result of such a move is again a filling in  $\mathcal{F}_{01}^{ne}(M, k)$ . In particular, we have to deal with the fact that under this identification all cells outside  $M$  are also filled with *elbow joints*, corresponding to *ones*. Of course, to determine the set of north-east chains we have to consider the original filling and the boundary of  $M$ , and *disregard* elbow joints outside.

Similar to the approach of Nantel Bergeron and Sara Billey we will consider two special fillings  $D_{top}(M, k)$  and  $D_{bot}(M, k)$ . These will turn out to be the maximal and the minimal element in the poset having elements  $\mathcal{F}_{01}^{ne}(M, k)$ , where one filling is smaller than another if it can

be obtained by applying chute moves to the latter. Figure 6 displays an example of the following construction:

**Definition 3.1.** Let  $M$  be a moon polyomino and  $k \geq 0$ . Then  $D_{top}(M, k) \in \mathcal{F}_{01}^{ne}(M, k)$  is obtained by putting ones into all cells that can be covered by any rectangle of size at most  $k \times k$ , which is completely contained in the moon polyomino, and that touches the boundary of  $M$  with its lower-left corner.

Similarly,  $D_{bot}(M, k) \in \mathcal{F}_{01}^{ne}(M, k)$  is obtained by putting ones into all cells that can be covered by any rectangle of size at most  $k \times k$ , which is completely contained in the moon polyomino, and that touches the boundary of  $M$  with its upper-right corner.

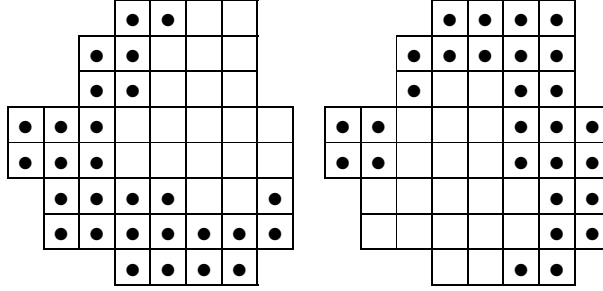


FIGURE 6. The special fillings  $D_{top}(M, k)$  and  $D_{bot}(M, k)$  for  $k = 2$  of a moon polyomino.

We can now state the main theorem of this article:

**Theorem 3.2.** Let  $M$  be a moon polyomino and  $k \geq 0$ . The set  $\mathcal{F}_{01}^{ne}(M, k, \mathbf{r})$  can be identified with the set of reduced pipe dreams  $\mathcal{RC}(w(M, k), \mathbf{r})$  having all crosses inside of  $M$  for some permutation  $w(M, k)$  depending only on  $M$  and  $k$ : replace zeros with crosses and all cells containing ones as well as all cells not in  $M$  with elbow joints.

More precisely, the set  $\mathcal{F}_{01}^{ne}(M, k)$  is an interval in the poset of reduced pipe dreams  $\mathcal{RC}(w(M, k))$  with maximal element  $D_{top}(M, k)$  and minimal element  $D_{bot}(M, k)$ .

As already remarked in the introduction various versions of this theorem were independently proved by various authors by various methods. The most general version is due to Luis Serrano and Christian Stump [15, Theorem 2.6], whose proof employs properties of subword complexes and who thus obtain additionally many interesting properties of the simplicial complex of 0-1-fillings.

The advantage of our approach using chute moves is the demonstration of the property that  $\mathcal{F}_{01}^{ne}(M, k)$  is in fact an interval in the bigger poset of reduced pipe dreams. In particular, if Conjecture 2.8 turns out to be true then  $\mathcal{F}_{01}^{ne}(M, k)$  is also a lattice. An illustration is given in Figure 4.

Let us first state a very basic property of chute moves as applied to fillings:

**Lemma 3.3.** *Let  $M$  be a moon polyomino. Chute moves and their inverses applied to a filling in  $\mathcal{F}_{01}^{ne}(M, k)$  produce another filling in  $\mathcal{F}_{01}^{ne}(M, k)$  whenever all zero entries remain in  $M$ .*

*Proof.* We only have to check that chain lengths are preserved, which is not hard.  $\square$

Most of what remains of this section is devoted to prove that there is precisely one filling in  $\mathcal{F}_{01}^{ne}(M, k)$  that does not admit a chute move such that the result is again in  $\mathcal{F}_{01}^{ne}(M, k)$ , namely  $D_{bot}(M, k)$ , and precisely one filling that does not admit an inverse chute move with the same property, namely  $D_{top}(M, k)$ .

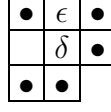
Although the strategy itself is actually very simple the details turn out to be quite delicate. Thus we split the proof into a few auxiliary lemmas. Let us fix  $k$ , a moon polyomino  $M$ , and a maximal filling  $D \in \mathcal{F}_{01}^{ne}(M, k)$  different from  $D_{bot}(M, k)$ . We will then explicitly locate a chutable rectangle. Throughout the proof maximality of the filling will play a crucial role. The first lemma is used to show that certain cells of the polyomino must be empty because otherwise the filling would contain a chain of length  $k + 1$ :

**Lemma 3.4** (Chain induction). *Consider a maximal filling of a moon polyomino. Let  $\epsilon$  be an empty cell such that all cells below  $\epsilon$  in the same column are empty too, except possibly those that are below the lowest cell of the column left of  $\epsilon$ . Assume that for each of these cells  $\delta$  there is a maximal chain for  $\delta$  strictly north-east of  $\delta$ . Then there is a maximal chain for  $\epsilon$  strictly north-east of  $\epsilon$ .*

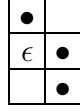
*Similarly, let  $\epsilon$  be an empty cell such that all cells left of  $\epsilon$  in the same row are empty too, except possibly those that are left of the left-most cell of the row below  $\epsilon$ . Assume that for each of these cells  $\delta$  there is a maximal chain for  $\delta$  strictly north-east of  $\delta$ . Then there is a maximal chain for  $\epsilon$  strictly north-east of  $\epsilon$ .*

*Remark 6.* Note that for the conclusion of Lemma 3.4 to hold we really have to assume that *all* cells below  $\epsilon$  are empty: in the maximal filling

for  $k = 1$

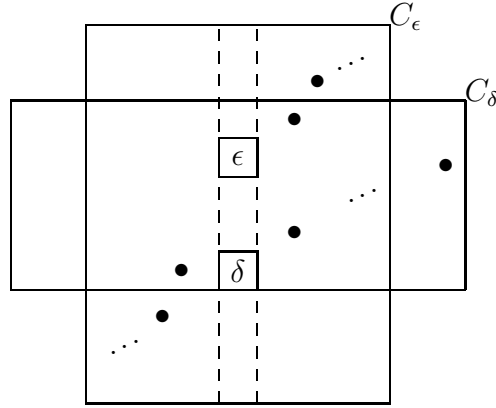


there is a maximal chain for  $\delta$  north-east of  $\delta$ , but no maximal chain for  $\epsilon$  north-east of  $\epsilon$ . The following example demonstrates that it is equally necessary that the filling is maximal:



*Proof.* Assume on the contrary that there is no maximal chain for  $\epsilon$  north-east of  $\epsilon$ . Consider a maximal chain  $C_\epsilon$  for  $\epsilon$  that has as many elements north-east of  $\epsilon$  as possible. Let  $\delta$  be the cell in the same column as  $\epsilon$ , below  $\epsilon$ , in the same row as the top entry of  $C_\epsilon$  which is south-east of  $\epsilon$ . By assumption, there is a maximal chain  $C_\delta$  for  $\delta$  north-east of  $\delta$ . We have to consider two cases:

If the widest rectangle containing  $C_\epsilon$  is not as wide as the smallest rectangle containing  $C_\delta$ , then the entry of  $C_\epsilon$  to the left of  $\delta$  would extend  $C_\delta$  to a  $(k + 1)$ -chain, which is not allowed:



If the smallest rectangle containing  $C_\epsilon$  is at least as wide as the widest rectangle containing  $C_\delta$ , then we obtain a maximal chain for  $\epsilon$  north-east of  $\epsilon$  by induction. Let  $c_\epsilon^1, c_\epsilon^2, \dots$  be the sequence of elements of  $C_\epsilon$  north-east of  $\epsilon$ , and  $c_\delta^1, c_\delta^2, \dots$  the sequence of elements of  $C_\delta$  north-east of  $\delta$ . We will show that  $c_\epsilon^i$  must be strictly north and weakly west of  $c_\delta^i$ , for all  $i$ . Thus, the elements  $c_\epsilon^1, c_\epsilon^2, \dots$  together with the elements of  $C_\delta$  outside the smallest rectangle containing  $C_\epsilon$  form a maximal chain for  $\epsilon$  north-east of  $\epsilon$ .

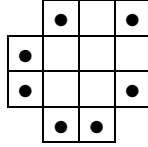
$c_\epsilon^1$  is strictly north of  $c_\delta^1$ , since otherwise  $C_\delta$  would be a maximal chain for  $\epsilon$ .  $c_\epsilon^1$  cannot be strictly east of  $c_\delta^1$ , since in this case  $c_\delta^1$  together with  $C_\epsilon$  would be a  $(k+1)$ -chain.

Suppose now that  $c_\epsilon^{i-1}$  is strictly north and weakly west of  $c_\delta^{i-1}$ .  $c_\delta^i$  cannot be strictly north-east of  $c_\epsilon^{i-1}$ , since this would yield a  $k$ -chain north-east of  $\epsilon$ .  $c_\delta^i$  must be strictly east of  $c_\epsilon^{i-1}$ , since  $c_\delta^i$  is strictly east of  $c_\delta^{i-1}$ , which in turn is weakly east of  $c_\epsilon^{i-1}$  by the induction hypothesis. Thus,  $c_\epsilon^{i-1}$  is weakly north and strictly west of  $c_\delta^i$ .

$c_\epsilon^i$  cannot be strictly north-east of  $c_\delta^i$ , since then the elements of  $C_\epsilon$  south-west of  $\epsilon$  together with the elements  $c_\delta^1, \dots, c_\delta^i$  and  $c_\epsilon^i, c_\epsilon^{i+1}, \dots$  would form a  $(k+1)$ -chain. Finally,  $c_\epsilon^i$  must be strictly north of  $c_\delta^i$ , since  $c_\epsilon^i$  is strictly north of  $c_\epsilon^{i-1}$ , which in turn is weakly north of  $c_\delta^i$ .  $\square$

**Lemma 3.5.** *Consider a maximal filling of a moon polyomino. Suppose that there is a rectangle with at least two columns and at least two rows completely contained in the polyomino, with all cells empty except the north-west, south-east and possibly the south-west corners. Then the south-west corner is indeed non-empty, i.e., the rectangle is chutable.*

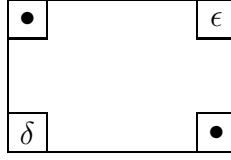
Note that we must insist that the south-west corner of the rectangle is part of the polyomino. Here is a maximal filling with  $k = 1$ , where the three cells in the south-west do not form a chutable rectangle, since the south-west corner is missing:



However, we can weaken this assumption in a different way:

**Lemma 3.6.** *Consider a maximal filling of a moon polyomino. Suppose that there is a rectangle with at least two columns and at least two rows such that all cells of its top row and its right column are contained in the polyomino. Assume furthermore that all cells of the rectangle that are in the polyomino are empty except the north-west, south-east and possibly the south-west corners. Finally, suppose that there is no maximal chain for the cell in the north-east corner strictly north east of it. Then the cell in the south-west corner is indeed in the polyomino and non-empty, i.e., the rectangle is chutable.*

*Proof of Lemma 3.5.* Suppose on the contrary that the cell in the south-west corner is empty, too. Then, the situation is as in the following picture:



Since the filling is maximal but the cells  $\delta$  and  $\epsilon$  are empty, there must be maximal chains for these cells. The corresponding rectangles must not cover any of the two cells containing ones, since that would imply the existence of a  $(k + 1)$ -chain. Thus, any maximal chain for  $\delta$  must be strictly south-west of  $\delta$ , and any maximal chain for  $\epsilon$  must be strictly north-east of  $\epsilon$ . Since the polyomino is intersection free, the top row of the rectangle containing the maximal chain for  $\epsilon$  is either contained in the bottom row of the rectangle containing the maximal chain for  $\delta$ , or vice versa. In both cases, we have a contradiction.  $\square$

The next lemma parallels the main Lemma 3.6 in the article by Nantel Bergeron and Sara Billey [1]:

**Lemma 3.7.** *Consider a maximal filling of a moon polyomino. Suppose that there is a cell  $\gamma$  containing a 1 with an empty cell  $\epsilon$  in the neighbouring cell to its right, such that there are at least as many cells above  $\gamma$  as above  $\epsilon$ . Then the filling contains a chutable rectangle.*

*Similarly, suppose that there is a cell  $\gamma$  containing a 1 with an empty cell  $\epsilon$  in the neighbouring cell below it, such that there are at least as many cells right of  $\gamma$  as right of  $\epsilon$ . Then the filling contains a chutable rectangle.*

*Proof.* Suppose that all of the cells in the column containing  $\epsilon$ , which are below  $\epsilon$  and weakly above the bottom cell of the column containing  $\gamma$ , are empty. Let  $\delta$  be the lowest cell in this region. There must then be a maximal chain for  $\delta$  that is north-east of  $\delta$ . By Lemma 3.4, we conclude that there is also a maximal chain for  $\epsilon$  north-east of  $\epsilon$ . However, then the 1 in the cell left of  $\epsilon$  together with this chain yields a  $(k + 1)$ -chain, since the rectangle containing the maximal chain for  $\epsilon$  extends by hypothesis to the column left of  $\epsilon$ .

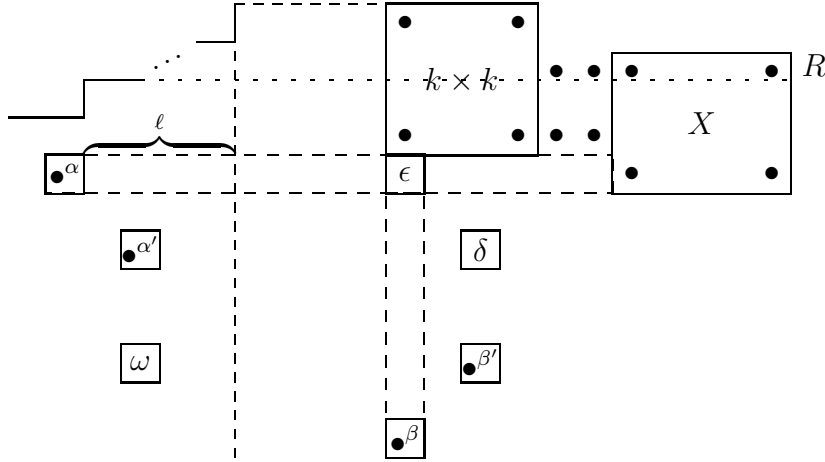
We can thus apply Lemma 3.5 to the following rectangle: the south-east corner being the top non-empty cell below  $\epsilon$ , and the north-west corner being the lowest cell containing a 1 in the column of  $\gamma$ , strictly above the chosen south-east corner.  $\square$

Finally, the main statement follows from a careful analysis of fillings different from  $D_{bot}(M, k)$ , repeatedly applying the previous lemmas to exclude obstructions to the existence of a chutable rectangle:

**Theorem 3.8.** *Any maximal filling other than  $D_{\text{bot}}(M, k)$  admits a chute move such that the result is again a filling of  $M$ . Any maximal filling other than  $D_{\text{top}}(M, k)$  admits an inverse chute move such that the result is again a filling of  $M$ .*

*Proof.* Suppose that all cells in the top-right quarter of  $M$  that contain a 1 in  $D_{\text{bot}}(M, k)$  also contain a 1 in the filling  $F$  at hand. It follows, that all cells that are empty in  $D_{\text{bot}}(M, k)$  are empty in  $F$ , too, because there is a maximal chain for each of them. Thus, in this case  $F = D_{\text{bot}}(M, k)$ .

Otherwise, consider the set of left-most cells in the top-right quarter, that contain a 1 in  $D_{\text{bot}}(M, k)$  but are empty in  $F$ , and among those the top cell,  $\epsilon$ . If its left or lower neighbour contains a 1, we can apply Lemma 3.7 and are done. Otherwise, we have to find a rectangle as in the hypothesis of Lemma 3.5. The difficulty in this undertaking is to prove that the lower left corner is indeed part of the polyomino. To ease the understanding of the argument, we will frequently refer to the following sketch:



By construction, there is a  $k \times k$  square filled with ones just above  $\epsilon$ , and there is no  $k$ -chain strictly north-east of  $\epsilon$ . This implies in particular that the top cell in the left-most column of the polyomino must be lower than the top row of the  $k \times k$  square, because otherwise there could not be any maximal chain for  $\epsilon$ .

By Lemma 3.4 there must therefore be a non-empty cell left of  $\epsilon$ , which we label  $\alpha$ , and a non-empty cell below  $\epsilon$ , which we label  $\beta$ . Note that there may be entries to the right of  $\epsilon$ , in the same row, which are non-empty. However, we can assume that to the right of the first such entry all other cells in this row are non-empty, too, because otherwise we could apply Lemma 3.7.



We can now construct a chutable rectangle: let  $\beta'$  be the top cell containing a 1 below an empty cell weakly to the right of  $\epsilon$ , and if there are several, the left-most. Also, let  $\alpha'$  be the lowest cell among the right-most containing a 1, which are weakly below  $\alpha$ , but strictly above  $\beta'$ . Let  $\delta$  be the cell in the same row as  $\alpha'$  and the same column as  $\beta'$ . Let  $\omega$  be the cell in the same column as  $\alpha'$  and the same row as  $\beta'$  – a priori we do not know however that  $\omega$  is a cell in the polyomino. We then apply Lemma 3.6 to the rectangle defined by  $\alpha'$  and the first non-empty cell to the right of  $\omega$ , in the same row.

To achieve our goal we show that there cannot be a maximal chain for  $\delta$  north-east of  $\delta$ . Suppose on the contrary that there is such a chain. At least its top-right element must be in a row (denoted  $R$  in the sketch) strictly above the top cell of the column containing  $\alpha'$ : otherwise,  $\alpha'$  together with this chain would form a  $(k+1)$ -chain. In the sketch, the non-empty cells that are implied are indicated by the rectangle denoted  $X$ , which must be of size  $k \times k$  at least.

If the size of the region to the right of  $\alpha$  indicated by  $\ell$  in the sketch is 0 we let  $\sigma$  be the bottom-left cell of a maximal chain for  $\epsilon$ . Otherwise, define  $\sigma$  to be the bottom-left cell of a maximal chain for the right neighbour of  $\alpha$ . In both cases,  $\sigma$  must be in a column weakly west of  $\alpha$  – in the latter case because there are fewer than  $k$  cells above the right neighbour of  $\alpha$ .

By intersection free-ness applied to the column containing  $\sigma$  and the columns containing cells of  $X$ , the latter columns must all extend at least down to the row containing  $\sigma$ . But in this case the maximal chain with bottom left cell  $\sigma$  can be extended to a  $(k+1)$ -chain using the cells in  $X$ .

We have shown that a maximal chain for  $\delta$  must have some elements south-east of  $\delta$ . We can now apply Lemma 3.6.  $\square$

*Proof of Theorem 3.2.* All pipe dreams in  $\mathcal{RC}(w)$  contained in  $M$  are maximal 0-1 fillings of  $M$ , since they can be generated by applying sequences of chute moves to  $D_{top}(M, k)$ .

Since we can apply chute moves to any maximal 0-1-filling of  $M$  except  $D_{bot}(M, k)$ , all such fillings arise in this fashion. (We have to remark here that in case the pipe dream associated to some filling would not be reduced, applying chute moves eventually exhibits that the filling was not maximal.) Together with Lemma 3.3, this implies that all fillings  $F_{01}^{ne}(M, k)$  have the same associated permutation.

Note that this procedure implies, as a by-product, that all maximal 0-1-fillings of  $M$  have the same number of entries equal to zero, *i.e.*, the simplicial complex of 0-1-fillings is pure.  $\square$

#### 4. APPLYING THE EDELMAN-GREENE CORRESPONDENCE

Using the identification described in the previous section, we can apply a correspondence due to Paul Edelman and Curtis Greene [2], that associates pairs of tableaux to reduced factorisations of permutations. This in turn will yield the desired bijective proof of Jakob Jonsson's result at least for stack polyominoes.

The main result of this section was obtained for Ferrers shapes earlier by Luis Serrano and Christian Stump [15] using the same proof strategy. For stack polyominoes the description of the  $P$ -tableau is different, thus we believe it is useful to repeat the arguments here.

The following theorem is a collection of results from Paul Edelman and Curtis Greene [2], Richard Stanley [16] and Alain Lascoux and Marcel-Paul Schützenberger [9], and describes properties of the *Edelman-Greene* correspondence:

**Theorem 4.1.** *Let  $w$  be a permutation and  $s_i$  be the elementary transposition  $(i, i + 1)$ . Consider pairs of words  $(u, v)$  of the same length  $\ell$ , such that  $s_{v_1}, s_{v_2}, \dots, s_{v_\ell}$  is a reduced factorisation of  $w$  and  $u_i \leq u_{i+1}$ , with  $u_i = u_{i+1}$  only if  $v_i > v_{i+1}$ .*

*There is a bijection between such pairs of words and pairs  $(P, Q)$  of Young tableaux of the same shape, such that  $P$  is column and row strict and whose reading word is a reduced factorisation of  $w$ , and such that the transpose of  $Q$  is semistandard. Moreover, if  $w$  is vexillary, i.e., 2143-avoiding, the tableau  $P$  is the same for all reduced factorisations of  $w$ .*

This correspondence can be defined via row insertion. We insert a letter  $x$  into row  $r$  of a tableau  $P$  whose last letter is different from  $x$  as follows: if  $x$  is (strictly) greater than all letters in row  $r$ , we just append  $x$  to row  $r$ . If row  $r$  contains both the letters  $x$  and  $x + 1$  we insert  $x + 1$  into row  $r + 1$ . Otherwise, let  $y$  be the smallest letter in row  $r$  that is strictly greater than  $x$ , replace  $y$  in row  $r$  by  $x$  and insert  $y$  into row  $r + 1$ .

We can now construct the pair of tableaux  $(P, Q) = (P_\ell, Q_\ell)$  from a pair of words  $(u, v)$  as in the statement of Theorem 4.1: let  $P_0$  and  $Q_0$  be empty tableaux. Insert the letter  $v_i$  into the first row of  $P_{i-1}$  to obtain  $P_i$ , and place the letter  $u_i$  into the cell of  $Q_i$  determined by the condition that  $P_i$  and  $Q_i$  have the same shape.

It turns out that the permutations associated to moon polyominoes are indeed vexillary:

**Proposition 4.2.** *For any moon-polyomino  $M$  and any  $k$  the permutation  $w(M, k)$  is vexillary.*

*Remark 7.* There are vexillary permutations which do not correspond to moon polyominoes. For example, the only two reduced pipe dreams for the permutation 4, 2, 5, 1, 3 are

$$\begin{array}{ccccc}
 & 1 & 2 & 3 & 4 & 5 \\
 4 & + & + & + & \bullet & \bullet \\
 2 & + & \bullet & + & \bullet & \\
 5 & + & \bullet & \bullet & & \\
 1 & \bullet & \bullet & & & \\
 3 & \bullet & & & & 
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccccc}
 & 1 & 2 & 3 & 4 & 5 \\
 4 & + & + & + & \bullet & \bullet \\
 2 & + & \bullet & \bullet & \bullet & \\
 5 & + & + & \bullet & & \\
 1 & \bullet & \bullet & & & \\
 3 & \bullet & & & & 
 \end{array}$$

*Proof.* It is sufficient to prove the claim for  $k = 0$ , since the empty cells in the filling  $D_{top}(M, k)$  for any  $k$  again form a moon polyomino. Thus, suppose that the permutation associated to  $M$  is not vexillary. Then we have indices  $i < j < k < \ell$  such that  $w(j) < w(i) < w(\ell) < w(k)$ . It follows that the pipes entering in columns  $i$  and  $j$  from above cross, and so do the two pipes entering in columns  $k$  and  $\ell$ , and thus correspond to cells of the moon polyomino. Since any two cells in the moon polyomino can be connected by a path of neighbouring cells changing direction at most once, there is a third cell where either the pipes entering from  $i$  and  $\ell$  or from  $j$  and  $k$  cross, which is impossible.  $\square$

**Theorem 4.3** (for Ferrers shapes, Luis Serrano and Christian Stump [15]). *Consider the set  $\mathcal{F}_{01}^{ne}(S, k, \mathbf{r})$ , where  $S$  is a stack polyomino. Let  $\mu_i$  be the number of cells the  $i^{\text{th}}$  row of  $S$  is indented to the right, and suppose that  $\mu_1 = \dots = \mu_k = \mu_{k+1} = 0$ .*

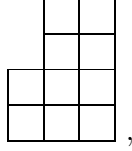
*Let  $u$  be the word  $1^{\mathbf{r}_1}, 2^{\mathbf{r}_2}, \dots$  and let  $v$  be the reduced factorisation of  $w$  associated to a given pipe dream. Then the Edelman-Greene correspondence applied to the pair of words  $(u, v)$  induces a bijection between  $\mathcal{F}_{01}^{ne}(S, k, \mathbf{r})$  and the set of pairs  $(P, Q)$  of Young tableaux satisfying the following conditions:*

- *the common shape of  $P$  and  $Q$  is the multiset of column heights of the empty cells in  $D_{top}(S, k)$ ,*
- *the first row of  $P$  equals  $(k+1, k+2+\mu_{k+2}, k+3+\mu_{k+3}, \dots)$ , and the entries in columns are consecutive,*
- *$Q$  has type  $\{1^{\mathbf{r}_1}, 2^{\mathbf{r}_2}, \dots\}$ , and entries in column  $i$  are at most  $i+k$ .*

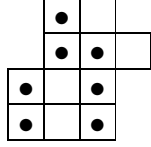
*Thus, the common shape of  $P$  and  $Q$  encodes the row lengths of  $S$ , the entries of the first row of  $P$  encode the left border of  $S$ , and the entries of  $Q$  encode the filling.*

*Remark 8.* In particular, this theorem implies an explicit bijection between the sets  $\mathcal{F}_{01}^{ne}(S_1, k, \mathbf{r})$  and  $\mathcal{F}_{01}^{ne}(S_2, k, \mathbf{r})$ , given that the multisets of column heights of  $S_1$  and  $S_2$  coincide.

Curiously, the most natural generalisation of the above theorem to moon polyominoes is not true. Namely, one may be tempted to replace the condition on  $Q$  by requiring that the entries of  $Q$  are between  $Q_{top}$  and  $Q_{bot}$  component-wise. However, this fails already for  $k = 1$  and the shape



with  $P = \begin{array}{|c|c|c|} \hline 3 & 4 & 5 \\ \hline 5 & & \\ \hline \end{array}$ ,  $Q_{top} = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 3 & & \\ \hline \end{array}$  and  $Q_{bot} = \begin{array}{|c|c|c|} \hline 2 & 3 & 4 \\ \hline 4 & & \\ \hline \end{array}$ . In this case, the tableau  $Q = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & & \\ \hline \end{array}$  has preimage



*Remark 9.* One might hope to prove Conjecture 2.8 by applying the Edelman-Greene correspondence, and checking that the poset is a lattice on the tableaux. However, at least for the natural component-wise order on tableaux, the correspondence is not order preserving, not even for the case of Ferrers shapes.

*Proof.* In view of Proposition 4.2, to obtain the tableau  $P$  it is enough to insert the reduced word given by the filling  $D_{top}(S, k)$  using the Edelman-Greene correspondence, which is not hard for stack polyominoes.

It remains to prove that the entries in column  $i$  of  $Q$  are at most  $i + k$  precisely if  $(u, v)$  comes from a filling in  $\mathcal{F}_{01}^{ne}(S, k)$ . To this end, observe that the shape of the first  $i$  columns of  $P$  equals the shape of the tableau obtained after inserting the pair of words  $((u_1, u_2, \dots, u_\ell), (v_1, v_2, \dots, v_\ell))$ , where  $\ell$  is such that  $u_\ell \leq k + i$  and  $u_{\ell+1} > k + i$ .

Namely, this is the case if and only if the first  $i + k + \mu_{i+k+1}$  positions of the permutation corresponding to  $(v_1, v_2, \dots, v_\ell)$  coincide with those of the permutation  $w$  corresponding to  $v$  itself, as can be seen by considering  $D_{top}(w)$ , whose empty cells form again a stack polyomino.

This in turn is equivalent to all letters  $v_m$  being at least  $k + i + 1 + \mu_{k+i+1}$  for  $m > \ell$ , i.e., whenever the corresponding empty cell of the filling occurs in a row below the  $(i + k)^{\text{th}}$  of  $S$ , and thus, when it is inside  $S$ .  $\square$

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I would like to acknowledge that Christian Stump provided a preliminary version of [17]. Luis Serrano and Christian Stump informed me privately that they were able to prove that all  $k$ -fillings of Ferrers shapes yield the same permutation  $w$ , however, their ideas would not work for stack polyominoes. I was thus motivated to attempt the more general case.

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